

TRANSFORMATION

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This article is dedicated to Professor K. Kannan Nambiar, my teacher and friend.

ABSTRACT. Transformation is the process of changing one signal into another. Transforms help us extract information from a signal.

This article explains how transforms are defined, and how they are calculated.

All transforms follow a common pattern. First the signal of interest is compared against each member of a set of building block functions. Every such comparison is made to yield a number. The set of numbers so generated is called a transform of the signal.

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1. Introduction

The purpose of transformation is to extract information from a signal, and to make hidden information visible.

When a gem is held up to light and turned, new facets become visible. Others, previously visible, are hidden. In a very similar way, a transform exposes some facets of a signal — even as it hides some others.

Different transforms reveal different kinds of information. The choice of the transform employed in a given situation is dictated by the sort of information we wish to unveil.

2. Prerequisites

This article assumes that the reader is familiar with elementary calculus.

3. Notation

By \mathbf{Z} , \mathbf{R} , and \mathbf{C} , we will mean, respectively, the set of integers (positive, negative, and zero), the set of real numbers, and the set of complex numbers.

By \mathbf{Z}_N we denote the set $\{0, 1, 2, \dots, (N - 1)\}$ of the first N integers.

If A is a set, then by $|A|$ we will mean the number of elements in A . $|A|$ is called the *cardinal number* of A .

The set of all function from a set A to a set B will be written B^A . If f is a function from A to B , $f : A \rightarrow B$, then $f \in B^A$. It should be admitted that it takes

some people a little while to get to like this notation. By way of some justification we point out that it is true that

$$(3.1) \quad |B^A| = |B|^{|A|}.$$

4. Signals

A *signal* is another name for a *function*. We will use these two words interchangeably.

Let us consider a few examples of the sort of signals that might be of interest to us. Let f be a function from the set \mathbf{R} of real numbers to the set \mathbf{R} . We attempt to picture f in Figure 1.

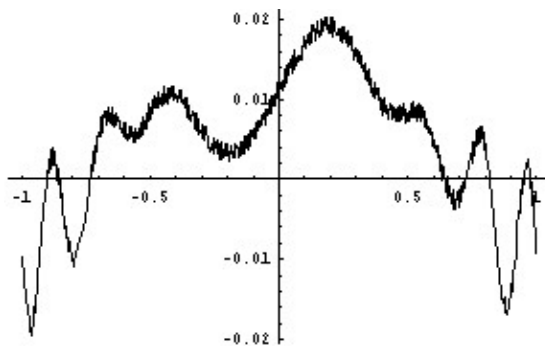


FIGURE 1. An analog signal from \mathbf{R} to \mathbf{R} .

It is impossible to describe a function in $\mathbf{R}^{\mathbf{R}}$ completely, unless a simple formula for it is known — which happens but rarely.

The sort of signal in Figure 1 is called an *analog signal*. It might represent, for example, the voltage at the terminals of a microphone.

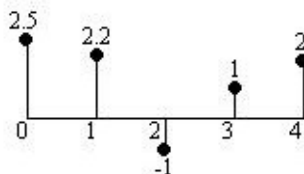
Often we are interested in analog signals that are not defined for all time. Thus, for example, we might find ourselves interested in a signal $g \in \mathbf{R}^{[0,1]}$ defined only on the closed interval $[0, 1]$ of the real line.

Signals in sets like $\mathbf{R}^{\mathbf{Z}}$ or $\mathbf{R}^{\mathbf{Z}^N}$ are very different from the (analog) signals considered above. Signals in $\mathbf{R}^{\mathbf{Z}}$ or $\mathbf{R}^{\mathbf{Z}^N}$ are called *discrete-time signals*. They are best pictured by way of “lollipop diagrams”. In Figure 2, for example, we draw the lollipop diagram of a signal in $\mathbf{R}^{\mathbf{Z}^5}$.

5. Building Blocks. An Example

A transformation is defined by a set of building blocks. The exact relationship between transforms and building blocks will be explored in the next section. In this section we’ll content ourselves with one example of a set of building blocks.

Let $\phi_1, \phi_2, \phi_3 \in \mathbf{R}^{\mathbf{R}}$ be three functions. We will think of the set $\Phi = \{\phi_1, \phi_2, \phi_3\}$ of these functions as a set of *atomic* functions, or as a set of *building blocks*. We will see how we might build new functions, or molecules, out of these building blocks.

FIGURE 2. A discrete-time signal defined from \mathbf{Z}_5 to \mathbf{R} .

For the sake of illustration we chose the functions ϕ_1, ϕ_2, ϕ_3 as in Figure 3. The choice of these particular building blocks is completely arbitrary. Useful transforms choose their building blocks with care — as we shall see.

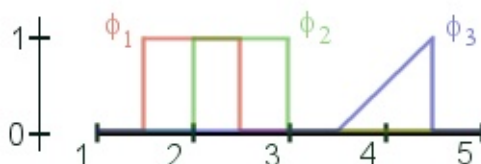


FIGURE 3. Three building block functions.

How might we build new signals with these three building blocks? What signals can we build? And what signals can we not?

Let c_1, c_2, c_3 be three real numbers. Define

$$(5.1) \quad f = c_1 \phi_1 + c_2 \phi_2 + c_3 \phi_3 = \sum_{n=1}^3 c_n \phi_n.$$

The function f is said to be a *linear combination* of the functions ϕ_1, ϕ_2, ϕ_3 . Every number c_n *scales* the corresponding function ϕ_n . Every number c_n quantifies the amount of the corresponding ϕ_n that goes into the construction of f .

The equation in (5.1) is also said to define a *decomposition* of f , because it shows how f breaks down in terms of the building blocks we chose.

The set of all linear combination of functions in a set Φ is said to constitute the *span* of Φ . Define

$$(5.2) \quad V = \text{span}(\Phi)$$

$$(5.3) \quad = \text{span}(\{\phi_1, \phi_2, \phi_3\})$$

$$(5.4) \quad \stackrel{\text{def}}{=} \left\{ g \in \mathbf{R}^{\mathbf{R}} \mid g = \sum_{n=1}^3 a_n \phi_n; a_n \in \mathbf{R} \right\}.$$

From (5.1) and (5.4), $f \in V$. We will call V a *function space*. Every element of the set V is a function from \mathbf{R} to \mathbf{R} .

In Figure 4 we see an example of a function g that *cannot* be built with the given building blocks. $g \notin V$.

Any function built with the elements of Φ must of necessity vanish in the interval $(3, 3.5)$. This is one of two reasons why $V \not\ni g$. The second reason is that the part of g that should conform to the shape of ϕ_3 does not.

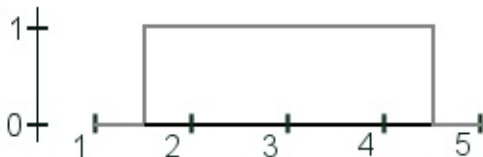


FIGURE 4. A function g that cannot be built with the three building blocks at our disposal. There is an infinity of functions that we can build with our chosen set of three building block functions. But there is also an infinity of functions that we cannot build.

We come face-to-face here with a situation that we encounter all the time: *the expressive power of any given set of building blocks is limited*. This is a simple but important lesson. To underscore it we draw the picture in Figure 5.

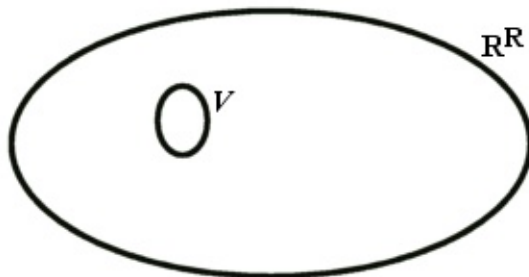


FIGURE 5. $V = \text{span}(\Phi)$ is a small subset of all functions from \mathbf{R} to \mathbf{R} .

6. Transforms

Given a knowledge of the set Φ of building blocks, and given a knowledge of the coefficients $\{c_n\}$, it is very easy to build up the corresponding function $f = \sum_n c_n \phi_n$.

For example, given the building blocks of the last section, and given that $c_1 = 1, c_2 = -0.5, c_3 = 0.5$, it is easy to see that $f = \sum_n c_n \phi_n$ must be as in Figure 6.

The reverse process: given f to find its coefficients c_n in the decomposition (5.1) is a little more involved — as we shall see.

However, suppose that we are able to decompose f in terms of the building block in the set Φ , as in equation (5.1); and let us designate by \hat{f} the set of the three coefficients c_1, c_2, c_3 that correspond to f ,

$$(6.1) \quad \hat{f} = (c_1, c_2, c_3),$$

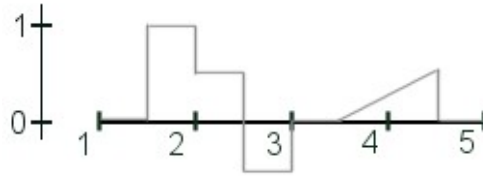


FIGURE 6. The function f built with the coefficients $c_1 = 1, c_2 = -0.5, c_3 = 0.5$ and the building blocks of the last section.

then \hat{f} is said to be a *transform* of f with respect to Φ .

Clearly \hat{f} is an ordered set of three elements. It is, in other words, a function from the set \mathbf{Z}_3 to \mathbf{R} .

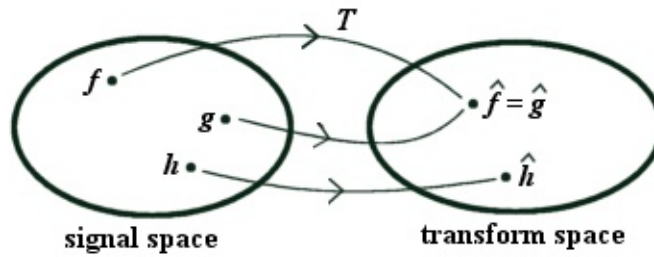


FIGURE 7. Transforms \hat{f}, \hat{g} , and \hat{h} of the signals f, g , and h , respectively. These are all calculated with respect to the set Φ of building blocks in Figure 3.

We see from Figure 7 that there may exist a signal $g \neq f$ such with $\hat{g} = \hat{f}$. An example of one such signal can be found in Figure 8.

In fact there exist an infinity of signals whose transforms coincide with \hat{f} .

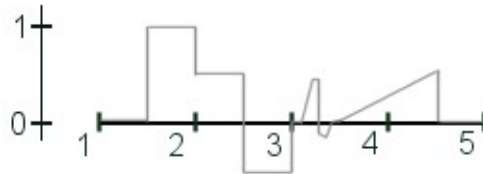


FIGURE 8. A signal $g \neq f$ whose image \hat{g} under transform T coincides with the image \hat{f} of f .

7. Building Blocks and Transforms

Every transform method, whether Fourier, Laplace, wavelet, or some other, differs in its choice of building blocks. Other than that the process of transformation is identical.

The building blocks of choice for the Fourier transform are complex exponentials. For the Laplace transform they are complex exponentials that decay exponentially in time. Wavelets allow many different choices of building blocks. So really there are many different transforms that are lumped within the term “wavelet transform”.

Apart from their choice of building blocks all transform methods are identical, in that they all start with a signal of interest, and they ask: How much of each building block is used in building the signal? The answer to this question is a whole bunch of numbers, like the coefficients c_n of our last section. This set of coefficients is called the *transform space image* of the signal, or briefly the transform of the signal.

The only difference between different transforms lies in their specific choice of building blocks.

[TO BE CONTINUED]

REFERENCES

- [1] Nicholas Young, *An Introduction to Hilbert Space*, Cambridge University Press, 1990.
- [2] Michael Unser, *Sampling — 50 Years After Shannon*, Proceedings of the IEEE, Vol. 88, No. 4, April 2000, pp. 569 - 587.

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